

# Topology Change and $\theta$ -Vacua in 2D Yang-Mills Theory

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We discuss the existence of  $\theta$ -vacua in pure Yang-Mills theory in two space-time dimensions. More precisely, a procedure is given which allows one to classify the distinct quantum theories possessing the same classical limit for an arbitrary connected gauge group  $G$  and compact space-time manifold  $M$  (possibly with boundary) possessing a special basepoint. For any such  $G$  and  $M$  it is shown that the above quantizations are in one-to-one correspondence with the irreducible unitary representations (IUR's) of  $\pi_1(G)$  if  $M$  is orientable, and with the IUR's of  $\pi_1(G)/2\pi_1(G)$  if  $M$  is nonorientable.

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Theories of interacting gauge and matter fields in two-dimensional space-time bear enough resemblance to their four-dimensional counterparts to make them interesting theoretical laboratories within which to investigate various qualitative effects, as well as test certain approximation schemes. However, in contrast to the situation in dimension three or more, there are no propagating gauge bosons in two dimensions. One may therefore expect that *pure* gauge theories in two dimensions are trivial. But if the two-dimensional space-time  $M$  has a nontrivial topology, then this is not quite true. For example, let's consider the case where  $M$  has the topology of a cylinder, the “space manifold” being simply a circle. Then the nontrivial fixed-time degrees of freedom are the path-ordered exponentials (or *holonomies*) of the gauge fields along a contour which winds once around the spatial circle, and the theory can be shown to be equivalent to the quantum mechanics of a free particle moving on the gauge group  $G$  [1][2]. (There are some subtleties here concerning boundary conditions which we will address later.) The radius  $R$  of the circle and the gauge coupling constant determine the parameters of this equivalent system. In particular, the volume of “space” (that is,  $G$ ) in the quantum mechanical model decreases as  $R$  increases. In the limit  $R \rightarrow \infty$  this volume tends to zero, causing all energy level differences to grow without bound and leaving us with a trivial theory as expected. Other topologically interesting space-times do not admit such a nice canonical decomposition into (space-manifold)  $\times$  (time). Consequently, pure gauge theories on these manifolds do not have an interpretation as ordinary quantum mechanical models as above, although they are still “less than field theories”. Indeed, they are all “almost topological” in the sense that the only property of the metric on  $M$  that the theory depends on is the area. Recently, the exact partition functions of these models have been obtained for an arbitrary space-time and gauge group [3]. These were then used to show that in the large- $N$  limit, any pure  $U(N)$ ,  $SU(N)$ ,  $SO(N)$ , or  $Sp(N)$  gauge theory on a closed two-dimensional space-time is equivalent to a closed string theory [4]. This holds for any value of the gauge coupling constant, except when space-time is the sphere  $S^2$  or the projective plane  $P^2$  in which case there is a third-order phase transition (from a trivial to a stringy phase) as one moves from weak to strong coupling [5]. These results lend support to the long-held belief that there is a stringy formulation of four-dimensional nonabelian gauge theories.

Another interesting feature that gauge theories in two and four dimensions share is the existence of  $\theta$ -vacua. That is, there exist in general numerous quantizations of the theory which possess the same classical limit. The name “ $\theta$ -vacua” derives from QCD

in ordinary four-dimensional Minkowski space, where the above quantum theories are labelled by an angle  $\theta$  and can be implemented by adding the well-known Pontryagin density (with coefficient  $\theta$ ) to the Lagrangian. A single angle also labels the quantizations of a  $U(1)$  gauge theory in two-dimensional Minkowski space (the *Schwinger model*) [6], where the topological term in the Lagrangian is now  $\frac{\theta}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu}$ . For *nonabelian* gauge theories in two-dimensional Minkowski space with a semi-simple gauge group, there is no longer a continuous parameter labelling the inequivalent quantum theories. There are now only a discrete number of choices available, parametrized by the distinct representations of the gauge group  $G$ . Moreover, there is no local term that can be added to the Lagrangian that can implement any of these choices. Instead, a multiplicative term in the quantum mechanical path integral must be inserted. More precisely, in order to obtain the quantization associated to a given representation  $D$  of  $G$ , one must multiply the path-integrand by the Wilson loop of the gauge field around the circle at space-time infinity, where the trace in the Wilson loop is evaluated in the representation  $D$  [7]. Not every distinct choice of  $D$  will yield a distinct quantum theory. The number of inequivalent theories depends on both the gauge group  $G$  and the matter content of the model. (In two-dimensional Minkowski space, which has the topology of the plane, some matter fields are necessary in order to obtain a nontrivial theory as noted earlier.) For example, if  $G = SU(n)$  and the model contains quarks in the fundamental representation, then all choices of  $D$  yield the same theory. If the quarks are instead in the adjoint representation, then there are  $n$  distinct quantizations (or vacua) [7]. In recent years, there have been numerous investigations into the properties of these multiple vacua in 2D gauge theories (see [8], and references therein).

All of these Minkowski space results, including the standard results in four dimensions, can be understood by considering the topology of the gauge theory configuration space (see [9], and references therein). In a pure gauge theory in  $(d+1)$ -dimensions with connected gauge group  $G$ , the classical configuration space consists of the set  $\mathcal{A}$  of all fixed-time, finite-energy gauge fields (in temporal gauge). However, we must identify any two elements of  $\mathcal{A}$  that differ by a static local gauge transformation which is trivial at spatial infinity. More precisely, the set of all such gauge transformations form an infinite-dimensional Lie group under pointwise multiplication. This *group of gauge transformations*, which we denote by  $\mathcal{G}$ , acts on  $\mathcal{A}$  in the usual way. The actual configuration space of the theory is the orbit space  $Q = \mathcal{A}/\mathcal{G}$ . The restriction that gauge transformations in  $\mathcal{G}$  are asymptotically trivial assures us that gauge fields differing by a nontrivial *global* gauge transformation are treated as distinct. It also makes the action of  $\mathcal{G}$  on  $\mathcal{A}$  free (that is, no gauge field is fixed under

any nontrivial element of  $\mathcal{G}$ ), so that  $Q$  is a smooth manifold. This boundary condition in effect compactifies the space manifold  $\mathbb{R}^d$  to the  $d$ -sphere  $S^d$ , which comes equipped with a special *basepoint* — the “point at infinity”. The wave functionals composing the Hilbert space of any given quantization of the system can be viewed as sections of a fixed complex vector bundle over  $Q$ . The quantum theories associated with the *flat* bundles all have the same classical limit, and are in one-to-one correspondence with the distinct (up to overall conjugation) unitary representations of the fundamental group  $\pi_1(Q)$ . It suffices to consider only the irreducible unitary representations (IUR’s) of  $\pi_1(Q)$  since all other quantizations can be easily obtained from these “irreducible” ones. In what follows, by a *quantization* we will mean a quantum theory constructed using one of these irreducible flat vector bundles over  $Q$ .

It is straightforward to show that  $\mathcal{A}$  is not, in general, a path-connected space. The path-components (or *soliton sectors*)  $\mathcal{A}_\alpha$  of  $\mathcal{A}$  are labelled by the elements  $\alpha \in \pi_{d-1}(G)$  — that is,  $\pi_0(\mathcal{A}) = \pi_{d-1}(G)$ . These different sectors correspond to topologically distinct behaviors of the gauge fields at spatial infinity. Since the elements of  $\mathcal{G}$  are trivial at spatial infinity, modding out by  $\mathcal{G}$  does not affect this classification. In other words, we also have  $\pi_0(Q) = \pi_{d-1}(G)$ . It can further be shown that each path-component of  $\mathcal{A}$  is contractible — that is, for each  $\alpha \in \pi_{d-1}(G)$  we have  $\pi_m(\mathcal{A}_\alpha) = \{e\}$  for all  $m \geq 1$ . This fact, combined with the freeness of the  $\mathcal{G}$  action and some standard results from algebraic topology, suffices to show that  $\pi_m(Q_\alpha) = \pi_{m-1}(\mathcal{G}) = \pi_{m+d-1}(G)$  for all  $\alpha \in \pi_{d-1}(G)$  and  $m \geq 1$ . In particular, we have that  $\pi_1(Q_\alpha) = \pi_d(G)$  for all  $\alpha$ . Thus, the inequivalent quantizations of the gauge theory (in any soliton sector) are labelled by the IUR’s of  $\pi_d(G)$ . When we add ordinary matter fields to the theory, the configuration space will clearly change. However, it can be shown that the homotopy groups of  $Q$ , in particular the fundamental group, are unaffected. The matter content of the theory *may* affect, however, the choice of gauge group  $G$  which should be used in the above analysis. For example, in a pure gauge theory the fields (which lie in the Lie algebra of  $SU(n)$ ) cannot detect the center of  $SU(n)$ , which is isomorphic to the cyclic group  $\mathbb{Z}_n$ . Therefore, the natural choice for  $G$  is the quotient group  $SU(n)/\mathbb{Z}_n$ . This result doesn’t change if we add adjoint matter to the theory since these fields are also unaffected by central elements. But the *fundamental representation* of  $SU(n)$  is faithful, so that the addition of fields transforming according to this representation leads us to choose  $G = SU(n)$ . Other choices of matter representations will yield, in general, different quotient groups of  $SU(n)$ .

Since  $\pi_3(G)$  is isomorphic to the additive group of the integers  $\mathbb{Z}$  when  $G$  is simple and nonabelian, we see that in three spatial dimensions there are a continuum of inequivalent quantizations of the corresponding gauge theories. More precisely, the IUR's of  $\mathbb{Z}$  are all one-dimensional and are labelled by an angle  $\theta$  — the IUR corresponding to a given  $\theta$  is given by  $n \rightarrow e^{in\theta}$  for all  $n \in \mathbb{Z}$ . These yield the well-known  $\theta$ -vacua discussed previously. By contrast, since  $\pi_3(U(1))$  is trivial there is no such phenomenon in (3+1)-dimensional QED. Also, the triviality of  $\pi_2(G)$  for all Lie groups implies that there are no  $\theta$ -vacua in (2+1)-dimensions. However, since  $\pi_1(U(1)) = \mathbb{Z}$ , we see the existence of  $\theta$ -vacua in (1+1)-dimensional QED as noted above. Simply-connected gauge groups ( $\pi_1(G) = \{e\}$ ) such as  $SU(n)$  do not lead to any such quantization ambiguities in 2D, while many other semi-simple gauge groups (such as  $SO(n)$  or  $SU(n)/\mathbb{Z}_n$ ) have a finite fundamental group ( $\pi_1(SO(n)) = \mathbb{Z}_2$  and  $\pi_1(SU(n)/\mathbb{Z}_n) = \mathbb{Z}_n$ ). These latter groups possess the discrete set of quantizations alluded to earlier.

The above results also hold in the case where the space manifold is not  $\mathbb{R}^d$ , but the solid  $d$ -dimensional ball  $B^d$  of some finite radius  $R$ . This is true so long as the boundary conditions imposed on the gauge fields and gauge transformations on the  $(d-1)$ -dimensional spherical boundary of  $B^d$  are similar to the corresponding conditions imposed at spatial infinity in the  $\mathbb{R}^d$  case — in effect compactifying  $B^d$  to  $S^d$  with a special basepoint.<sup>1</sup> For  $d=1$ , space is simply a line segment which is compactified to a circle. The basepoint  $y_0$  of this circle represents the two endpoints of the line segment which are equated by the boundary conditions. Note that the holonomy of any static gauge field  $A$  around this spatial circle (based at  $y_0$ ) is invariant under the elements of  $\mathcal{G}$ , which are required to be trivial at  $y_0$ . Moreover, specifying this holonomy is equivalent to specifying the gauge equivalence class of  $A$  in the configuration space  $Q$ . Since these holonomies can be any element of the gauge group  $G$ , we see that  $Q$  is homeomorphic to the group manifold of  $G$ . (In particular, we recover  $\pi_0(Q) = \pi_0(G)$  and  $\pi_1(Q) = \pi_1(G)$ .) By contrast,  $Q$  is infinite-dimensional for all  $d > 1$  since here we have a true field theory — that is, there are propagating gauge bosons.

There is a subtlety here concerning boundary conditions. Namely, the way in which we obtained a cylindrical space-time above was to compactify space from a line segment to a circle. But we may also wish to consider the case where space is simply a circle at the outset,

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<sup>1</sup> Even though the configuration spaces have the same topology in these two cases, the *dynamics* of the two theories can be quite different. They must, however, coincide as  $R \rightarrow \infty$ .

and not just as a result of boundary conditions. The difference in these two situations lies in the existence of a basepoint. Saying that space is *a priori* compact means that there is no special point on the circle chosen by the physics — all points are on the same footing. As a result, there is no natural place to require the elements of  $\mathcal{G}$  to be trivial. If we simply choose a random point at which to do so, this will violate our initial assumption about the nature of the space manifold. In other words, we will be changing the physics. But if we do not have such a point, then we are forced to include *all* static gauge transformations in  $\mathcal{G}$ , even global ones. Since holonomies conjugate under global gauge transformations, the configuration space  $Q = G$  obtained in the “pointed” case above is then replaced by the orbit space  $G/G$ , where  $G$  acts on itself by conjugation. That is,  $Q$  is now the space of *conjugacy classes* in  $G$ . (For instance, the space of conjugacy classes of  $G = SO(3)$  is homeomorphic to a line segment.) Understanding the inequivalent quantizations of these theories is a bit more involved [2]. However, there remains a subclass of quantizations which are labelled by the unitary representations of  $\pi_1(G)$ . In what follows, we will always assume a special basepoint and avoid these complications.

The above canonical methods generalize to any space-time which is homeomorphic to  $\Sigma \times I$ , where  $\Sigma$  is a space-like manifold of dimension  $d \geq 1$  and  $I = [0, 1]$  is the unit interval on the real line. One simply replaces  $\mathcal{A}$  by the space of all static gauge fields on  $\Sigma$ , and  $\mathcal{G}$  by the associated group of static gauge transformations (both satisfying restrictions appropriate to the topology of  $\Sigma$ ). The configuration space  $Q(\Sigma)$  is again the corresponding orbit space. The inequivalent quantizations of the theory are still labelled by the IUR’s of  $\pi_1(Q(\Sigma))$ . However, the computation of  $\pi_1(Q(\Sigma))$  is somewhat more involved for a general  $\Sigma$ . In particular, the result is no longer simply  $\pi_d(G)$  in general [10].

We now turn to the main purpose of this paper, which is to determine the inequivalent quantizations of a pure Yang-Mills theory on a *general* two-dimensional compact space-time  $M$  (possibly with boundary). We are particularly interested in cases where there is no nonsingular global slicing of  $M$  into space-like manifolds, since the canonical techniques described earlier do not apply here. For example,  $M$  can be the two-sphere  $S^2$  (or any closed surface other than the torus), or have the “pair of pants” topology shown in Fig.1. Such a situation is often said to involve *topology change* since any attempt to foliate  $M$  into space-like pieces will necessarily have special slices where space is not a manifold (there will be singular regions in space). Immediately before and immediately after these special time values the spatial slices *will* be manifolds in general, but their topologies may be quite different. In the example of Fig.1, the spatial topology can be viewed as

changing from that of a circle to that of a disjoint union of two circles through a singular space having the topology of a figure eight. Since fixed-time degrees of freedom are no longer well-defined globally, we must find a covariant analog of the canonical configuration space  $Q(\Sigma)$  above. We will use the *space of histories* which differs from  $Q(\Sigma)$  essentially by replacing  $\Sigma$  by the full space-time manifold  $M$ . More precisely,  $\mathcal{A}$  is replaced by the set  $\mathcal{A}^H$  of all space-time gauge fields, and  $\mathcal{G}$  is replaced by the group  $\mathcal{G}^H$  of all space-time gauge transformations. Again, the elements of both  $\mathcal{A}^H$  and  $\mathcal{G}^H$  are assumed to satisfy restrictions appropriate to the topology of  $M$ . The space of histories is then the orbit space  $Q^H(M) = \mathcal{A}^H/\mathcal{G}^H$ . However, since “time” has already been folded into the definition of  $Q^H(M)$ , the analog of fundamental group  $\pi_1(Q(\Sigma))$  is the set  $\pi_0(Q^H(M))$  which labels the path-components (or *instanton sectors*) of  $Q^H(M)$ . A quantization of the theory is then associated with some map  $\chi$  from  $\pi_0(Q^H(M))$  to a unitary matrix group. Since  $\pi_0(Q^H(M))$  is not endowed with a natural group structure in general, it no longer makes sense to require this map to be a homomorphism. But there *are* certain restrictions on  $\chi$ . First, if  $M$  is globally of the form  $\Sigma \times I$  with  $\Sigma$  a space-like manifold (that is, there is no topology change), then it is easy to show that  $\pi_0(Q^H(\Sigma \times I))$  is in one-to-one correspondence with  $\pi_1(Q(\Sigma))$ . In order to recover the canonical results, we must then consider only those  $\chi$ ’s which yield representations of  $\pi_1(Q(\Sigma))$  under any one such correspondence. All other  $\chi$ ’s are to be discarded.<sup>2</sup> More generally, if there is some region of  $M$  of the form  $\Sigma \times I$ , then we can always find a continuous map  $\phi : Q^H(\Sigma \times I) \rightarrow Q^H(M)$  which uses some fixed prescription to extend any history on  $\Sigma \times I$  to all of  $M$ . In turn, this will induce a map  $\phi_* : \pi_0(Q^H(\Sigma \times I)) \rightarrow \pi_0(Q^H(M))$  which relates local information about  $\pi_0(Q^H)$  from such a canonical region to the corresponding global information obtained utilizing the full structure of  $M$ . Using the previous result, we may rewrite this as  $\phi_* : \pi_1(Q(\Sigma)) \rightarrow \pi_0(Q^H(M))$ . Any quantization of the full theory must yield an allowed quantization in each such local region. Therefore, we should only consider  $\chi$ ’s such that for each canonical region of  $M$  as above the composite map  $\chi \circ \phi_*$  is a homomorphism (for some fixed prescription for constructing  $\phi$  and some fixed one-to-one correspondence between  $\pi_0(Q^H(\Sigma \times I))$  and  $\pi_1(Q(\Sigma))$ ). In other words, these  $\chi$ ’s label the inequivalent quantizations. We can still identify the “irreducible” quantizations from which all others

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<sup>2</sup> Changing the one-to-one correspondence  $\eta : \pi_0(Q^H(\Sigma \times I)) \rightarrow \pi_1(Q(\Sigma))$  which is used in this procedure will, in general, change the set of allowed  $\chi$ ’s. However, by construction, this set remains in one-to-one correspondence with the representations of  $\pi_1(Q(\Sigma))$  for any choice of  $\eta$ .

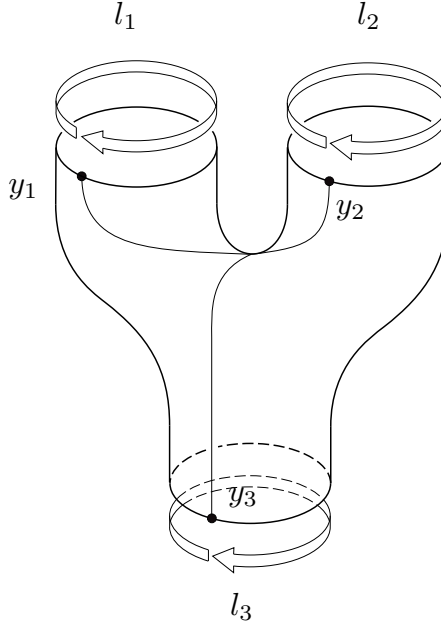


Fig. 1: The “pair of pants” space-time  $P$ . The loops  $l_1, l_2$ , and  $l_3$  (based at the points  $y_1, y_2$  and  $y_3$ ) are used to define the boundary holonomies  $g_1, g_2$ , and  $g_3$ . The “history” of the basepoints is also indicated. It consists of a network having a “Y” shape.

can be built as those for which the matrices  $\chi(\pi_0(Q^H(M)))$  constitute an irreducible set. A few examples should help clarify these issues.

First, let’s return to the case of a pure gauge theory, with connected gauge group  $G$ , defined on a space-time cylinder  $C = S^1 \times I$  where the spatial circle has a special basepoint. As noted earlier, the canonical configuration space  $Q(S^1)$  is homeomorphic to the group manifold of  $G$ . Even though this theory contains no topology change, let us repeat the analysis using the space of histories  $Q^H(C)$ . By following the basepoint of the spatial circle through time, we obtain a curve in space-time starting at one end of the cylinder and ending at the other. We parametrize this curve by  $y_t, t \in [0, 1]$ . At each such time  $t$  the holonomy of a gauge field configuration in  $\mathcal{A}^H$  around the spatial circle (based at  $y_t$ ) yields an element  $g(t) \in G$ . Since we again wish to treat fields differing by a global gauge transformation as distinct, we require each gauge transformation in  $\mathcal{G}^H$  to be trivial at  $y_t$  for all  $t$ .<sup>3</sup> Each of the above holonomies  $g(t)$  is then invariant under the elements of  $\mathcal{G}^H$ ,

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<sup>3</sup> Actually, we only need to require this at any one time. However, our stronger assumption simplifies the analysis without changing the final result. More on this later.



and when taken together contain the same information as does the  $\mathcal{G}^H$  equivalence class of the underlying space-time gauge field. We must also fix the boundary holonomies on the two circular ends of  $C$  at  $t = 0$  and  $t = 1$ . We take these to be  $g_1$  and  $g_2$  respectively. Thus,  $Q^H(C)$  can be viewed as the space of all maps  $g : I \rightarrow G$  satisfying  $g(0) = g_1$  and  $g(1) = g_2$ . We can also view the theory as a (0+1)-dimensional “nonlinear sigma model” with target space  $G$  — or equivalently, the quantum mechanics of a single particle on  $G$  as noted above. Since  $Q^H(C)$  is nothing but the set of all continuous paths in  $G$  between  $g_1$  and  $g_2$ , we also see that  $\pi_0(Q^H(C))$  is in one-to-one correspondence with  $\pi_1(G)$  (for any choice of  $g_1$  and  $g_2$ ). Recall that we have previously shown  $\pi_1(Q(S^1)) = \pi_1(G)$ , so that we indeed obtain the same results in the new covariant approach as we did in the canonical picture described earlier.

What happens if we now change the space-time manifold from the cylinder to the manifold  $P$  shown in Fig.1? Near each of the three ends of  $P$ , space-time appears again to be simply a cylinder. Therefore, restricting ourselves to any one of these regions we should find the same quantization ambiguities labelled by the IUR’s of  $\pi_1(G)$  as above. The question now becomes: Does anything interesting occur when we attempt to paste these three regions together? That is, what labels the inequivalent quantizations on the full space-time? If we follow the spatial basepoint through “time” in this case, we no longer get a curve homeomorphic to the unit interval. Instead we obtain something akin to the letter “Y”. In other words, the two distinct basepoints on the two upper legs of space-time merge at the singular junction and become the basepoint of the lower leg. We require the elements of  $\mathcal{G}^H$  to be trivial along this entire “basepoint network”. The elements of  $Q^H(P)$  are then maps from this “Y” into  $G$ . Each such map, when evaluated at a point  $y \in Y$ , represents the holonomy (based at  $y$ ) of the gauge field around the spatial circle which contains  $y$ . There is, however, one exception to this. If  $y$  is the junction point, then the holonomy is evaluated along the entire spatial figure-eight. Each element of  $Q^H(P)$  is subject to boundary conditions which fix the values of the holonomy at all three ends. There is also one additional constraint — an extra condition on the holonomies at the junction. In order to determine this condition, imagine starting with a loop going once around the spatial circle at each of the three ends of  $P$ . Now move each of these loops toward the singular time value. It is clear that the loop starting on the lower leg of space-time becomes the composite of the other two loops at this special time slice. This means that the lower leg holonomy must equal the product of the two upper leg holonomies in the limit as we approach the junction of the “Y”. We may thus view  $Q^H(P)$  as follows.

First, consider a single copy of the gauge group  $G$  marked with three special points  $g_1$ ,  $g_2$  and  $g_3$ . These special elements of  $G$  represent the fixed holonomies of the gauge fields around the loops  $\ell_1$ ,  $\ell_2$  and  $\ell_3$  at the ends of  $P$  (see Fig.1). An element of  $Q^H(P)$  then corresponds to a choice of three paths in  $G$  emanating from  $g_1$ ,  $g_2$  and  $g_3$  and ending at some points  $h_1$ ,  $h_2$  and  $h_3$  respectively, subject to the constraint  $h_3 = h_1 h_2$ .

Now that we have a simple picture of  $Q^H(P)$  at our disposal, we can attempt to compute  $\pi_0(Q^H(P))$  and use this to classify the inequivalent quantizations of the system. Our claim is that the result is identical to that obtained above for  $Q^H(C)$  — that is,  $\pi_0(Q^H(P)) = \pi_1(G)$  independent of the choice of boundary holonomies on  $P$ , and the inequivalent quantizations are labelled by the IUR's of  $\pi_1(G)$ . In other words, the introduction of topology change has not affected the number of distinct quantum theories. In order to prove this, we will need to relate the spaces  $Q^H(P)$  and  $Q^H(C)$ . But first we must fix our boundary holonomies. On  $P$  we stick to the generic choices  $g_1$ ,  $g_2$  and  $g_3$  discussed previously. On  $C$  we fix the gauge field holonomies to be  $g_3$  on the “bottom” end of the cylinder, and  $g_1 g_2$  on the top. (The reason for this funny choice will be made clear momentarily, but remember that  $\pi_0(Q^H(C)) = \pi_1(G)$  for any choice of boundary holonomies.) We will now define a continuous map  $\phi : Q^H(C) \rightarrow Q^H(P)$ . If  $q_c \in Q^H(C)$ , we let  $\phi(q_c)$  be the history in  $Q^H(P)$  which is equal to  $q_c$  on the lower leg of the “Y” (corresponding to the lower cylindrical leg of  $P$ ), and is constant and equal to  $g_1$  (respectively,  $g_2$ ) on the upper left (respectively, right) leg. Thus,  $\phi(q_c)$  only changes with time in the period before the singularity. (Note that the constraint on the holonomies at the junction led to our choice of boundary conditions on  $C$ .) Clearly,  $\phi$  is a one-to-one map. That is,  $Q^H(C)$  is homeomorphic to the subspace  $\phi(Q^H(C))$  of  $Q^H(P)$ . We will now show that each history in  $Q^H(P)$  can be continuously deformed into this subspace. As noted earlier, any  $q_p \in Q^H(P)$  may be viewed as a set of three paths in  $G$  starting at  $g_1$ ,  $g_2$  and  $g_3$  and ending at some points  $h_1$ ,  $h_2$  and  $h_3$  respectively. These paths correspond to the holonomies on the three legs of  $P$  and must satisfy the junction constraint  $h_3 = h_1 h_2$ . We will first deform the path starting at  $g_1$  back along itself until it simply remains at the point  $g_1$ . While doing this we will keep the path starting at  $g_2$  fixed, so that in order to meet the junction constraint the path starting at  $g_3$  must simultaneously deform. When we are done, this path will end at  $g_1 h_2$ . We now do the same deformation of the path starting at  $g_2$ , keeping the (constant) path at  $g_1$  fixed, which again causes the path starting at  $g_3$  to change. What we are left with when we are finished are two constant paths, one at  $g_1$  and one at  $g_2$ , along with a path which starts at  $g_3$  and ends at  $g_1 g_2$ . That is, the gauge field

holonomy is constant (and equal to the appropriate boundary value) on each of the two upper legs of  $P$ , and only changes nontrivially on the lower leg. This final history, which we denote by  $\tilde{q}_p$ , lies in the subspace  $\phi(Q^H(C))$ . Since our initial history  $q_p$  was generic, we have shown that each path-component of  $Q^H(P)$  contains *at least one* path-component of  $\phi(Q^H(C))$ . In order to demonstrate that  $\pi_0(Q^H(P)) = \pi_0(Q^H(C))$ , it remains to be shown that each path-component of  $Q^H(P)$  contains *only one* path-component of  $\phi(Q^H(C))$ . To see this, note that a small change in the initial history  $q_p$  will lead to a correspondingly small change in the final history  $\tilde{q}_p$ . Therefore, moving  $q_p$  around its path-component in  $Q^H(P)$  simply moves  $\tilde{q}_p$  around its corresponding path-component in  $\phi(Q^H(C))$ . Now suppose that a given component of  $Q^H(P)$  contains two components of  $\phi(Q^H(C))$ . If  $q_p$  lies in either of these two components, the above deformation process will yield  $\tilde{q}_p = q_p$  since  $q_p$  already has constant holonomies on the two upper legs of  $P$ . But we can move  $q_p$  continuously through  $Q^H(P)$  from one component of  $\phi(Q^H(C))$  to the other. This implies that  $\tilde{q}_p$  also moves from one component of  $\phi(Q^H(C))$  to the other, contradicting the earlier result that continuous changes in  $q_p$  cannot change the the path-component of  $\tilde{q}_p$ . Hence, each path-component of  $Q^H(P)$  contains exactly one path-component of  $\phi(Q^H(C))$ , and the map  $\phi_* : \pi_0(Q^H(C)) \rightarrow \pi_0(Q^H(P))$  is a one-to-one correspondence. In other words,  $\pi_0(Q^H(P)) = \pi_1(G)$ .

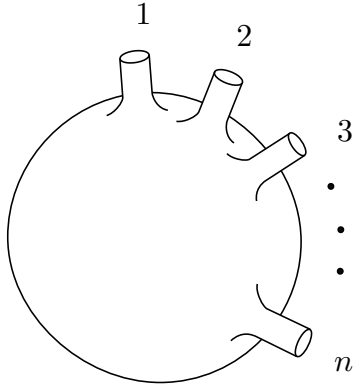


Fig. 2: The space-time  $X_n$ .

It is easy to demonstrate that this result still holds if we add additional legs to the manifold  $P$ . More precisely, let  $X_n$  denote a space-time with  $n$  cylindrical legs and  $n$  circular ends as shown in Fig.2. Both of the previous cases considered correspond to special

values of  $n$  — namely,  $X_2 = C$  and  $X_3 = P$ . We can always choose a time-slicing of  $X_n$  such that the associated basepoint network consists of a single junction with  $n$  attached legs. We continue to require the elements of  $\mathcal{G}^H$  to be trivial along this entire network. An element of  $Q^H(X_n)$  can then be viewed as a set of  $n$  paths in  $G$  (representing the holonomies on each of the  $n$  legs of  $X_n$ ) starting at the points  $g_1, g_2, \dots, g_n$  (representing the  $n$  boundary holonomies) and ending at some values  $h_1, h_2, \dots, h_n$  (representing the  $n$  holonomies at the junction). If we choose the  $n$ -th leg of  $X_n$  as “incoming” and the rest as “outgoing”, then there is also the junction constraint  $h_n = h_1 \cdots h_{n-1}$ . As before, we can continuously deform any such history so that the holonomies are constant on every leg except (say) the incoming one. That is, at the end of the deformation we have one path in  $G$  starting at  $g_n$  and ending at the product  $g_1 g_2 \cdots g_{n-1}$ , and  $n - 1$  constant paths at the points  $g_1, \dots, g_{n-1}$ . The same reasoning which we used to relate the  $n = 2$  and  $n = 3$  cases then shows that the spaces  $Q^H(X_n)$ ,  $n \geq 2$ , all have the same  $\pi_0$ . Note that one can obtain the space  $X_{n-1}$  by simply “closing off” one of the boundaries in  $X_n$ . Similarly, if we choose any one of the boundary holonomies on  $X_n$  to be the identity element of  $G$ , then the gauge theory on  $X_n$  becomes equivalent to a gauge theory on  $X_{n-1}$ . In this way we see that  $\pi_0(X_1)$  ( $X_1$  is a disk) and  $\pi_0(X_0)$  ( $X_0$  is a sphere) are the same as  $\pi_0(X_n)$ ,  $n \geq 2$ . The theories on  $X_1$  and  $X_0$  simply correspond to a special choice of boundary conditions in the higher  $n$  cases.

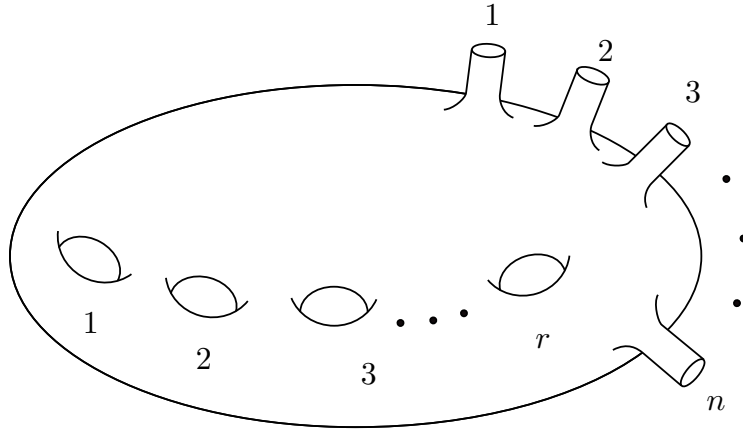


Fig. 3: The space-time  $X_{n,r}$ .

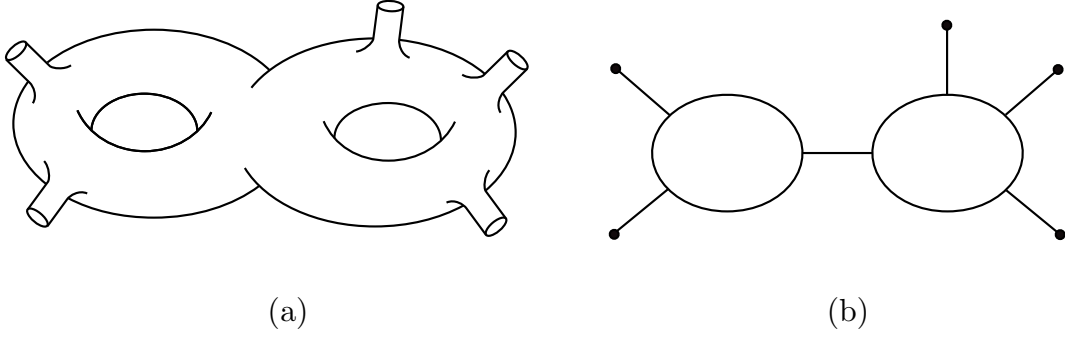


Fig. 4: (a) shows the space-time  $X_{5,2}$ . The basepoint network associated with a particular time-slicing of  $X_{5,2}$  is illustrated in (b).

More generally, we may add to  $X_n$  an arbitrary number of “handles” as in Fig.3. We denote this space-time with  $n \geq 0$  boundaries and  $r \geq 0$  handles (or *genus*  $r$ ) by  $X_{n,r}$ . (Note that  $X_{n,0} = X_n$ .) In any particular time-slicing, the space of histories  $Q^H(X_{n,r})$  will now consist of a certain class of maps from a complicated one-dimensional basepoint network into the gauge group  $G$ . This network will have  $n$  external legs,  $r$  internal “loops” and numerous junctions. (As an example, a possible basepoint network for  $X_{5,2}$  is shown in Fig.4.) As usual, there will be a constraint on the product of the holonomies at each junction, and we have required the elements of  $\mathcal{G}^H$  to be trivial along the entire network. The computation of  $\pi_0(Q^H(X_{n,r}))$  is similar to the genus zero case above, but with one additional subtlety. We outline the procedure below. First, given any  $q_c \in Q^H(C)$ , we can still find a history in  $Q^H(X_{n,r})$  which is equal to  $q_c$  on one external cylindrical leg of  $X_{n,r}$  and has constant holonomy on each other cylindrical portion (both internal and external). However, the values of these constants are no longer completely fixed by the combination of the boundary holonomies and the junction constraints. There will be one unconstrained constant for each of the  $r$  handles — that is, one for each internal loop in the corresponding basepoint network. (These are similar to unconstrained loop momenta in a Feynman diagram.) If we want a continuous and one-to-one map  $\phi : Q^H(C) \rightarrow Q^H(X_{n,r})$  as we had in the  $r = 0$  case, then an arbitrary choice must be made for these  $r$  “loop holonomies”. However, once these holonomies are fixed we can no longer give a well-defined, continuous procedure for deforming an arbitrary history in  $Q^H(X_{n,r})$  into the subspace  $\phi(Q^H(C))$ , since each history will have its own natural values for these constants. But there is a continuous and one-to-one map  $f : Q^H(C) \times G^r \rightarrow Q^H(X_{n,r})$ , where  $G^r = G \times \cdots \times G$  ( $r$  times), which avoids this problem. We simply include the unconstrained loop holonomies as extra degrees of freedom in the domain of the map. That is, if we specify a cylindrical

history in  $Q^H(C)$  along with  $r$  elements of  $G$  (representing the loop holonomies), then  $f$  gives us a history on  $X_{n,r}$  whose holonomy only changes on one external leg. In a manner similar to the  $r = 0$  case, we can now give a well-defined procedure for deforming any element of  $Q^H(X_{n,r})$  into the subspace defined by the image of  $f$ , and use this to show that  $\pi_0(Q^H(X_{n,r}))$  is in one-to-one correspondence with  $\pi_0(Q^H(C) \times G^r)$ . As  $G$  is assumed connected ( $\pi_0(G) = \{e\}$ ), this yields  $\pi_0(X_{n,r}) = \pi_0(Q^H(C)) = \pi_1(G)$ . Note that this result also applies to the closed surfaces  $X_{0,r}$  since the gauge theory on such a surface can be obtained from the theory on  $X_{n,r}$  (for any  $n \geq 1$ ) by choosing the boundary holonomies to be trivial. Since any compact, orientable surface is homeomorphic to some  $X_{n,r}$ , we have shown:

*For a two-dimensional pure Yang-Mills theory with (connected) gauge group  $G$  defined on a compact, orientable space-time  $M$ ,  $\pi_0(Q^H(M)) = \pi_1(G)$  independent of the topology of  $M$ .*

We can still view this correspondence as being induced by a map  $\phi : Q^H(C) \rightarrow Q^H(X_{n,r})$  which is obtained from  $f : Q^H(C) \times G^r \rightarrow Q^H(X_{n,r})$  by making some fixed choice of the  $r$  loop holonomies. That is,  $\phi_* : \pi_0(Q^H(C)) \rightarrow \pi_0(Q^H(X_{n,r}))$  is one-to-one and onto. Since  $C$  may be viewed as the cylindrical leg of  $X_{n,r}$  onto which we are deforming all the nontrivial holonomy changes, and any quantization of the full gauge theory must yield an allowed quantization in this local region, we see that the inequivalent quantizations of the gauge theory on  $X_{n,r}$  (for any  $n$  and  $r$ ) are labelled by the IUR's of  $\pi_1(G)$ .

It is worth mentioning that although we have always used a particular time-slicing of  $M$  in order to perform our analysis, the above result holds for *any* such choice. That is,  $\pi_0(Q^H(M))$  is independent of the choice of time-slicing. Our result in the  $r > 0$  case *does*, however, depend critically on the assumption that  $G$  is connected. For gauge groups with more than one path-component, such as  $O(N)$ , a more careful analysis yields  $\pi_0(Q^H(X_{n,r})) = \pi_1(G) \times \pi_0(G)^{2r}$  (when the elements of  $\mathcal{G}^H$  are required to be trivial only at a single point on  $X_{n,r}$ ), and  $\pi_0(Q^H(X_{n,r})) = \pi_1(G) \times \pi_0(G)^r$  (when the elements of  $\mathcal{G}^H$  are required to be trivial along the entire basepoint network). Note that when  $G$  is connected we obtain  $\pi_0(Q^H(X_{n,r})) = \pi_1(G)$  for either of these two restrictions on the gauge transformations in  $\mathcal{G}^H$ . Therefore, as alluded to earlier, the results of this paper remain unchanged if we assume that space-time has only a *single* special basepoint rather than a network of such points. But our results *do* change if there is not at least one special

point. More specifically, if  $M$  is *a priori* compact then there is no natural place at which we can require the gauge transformations in  $\mathcal{G}^H$  to be trivial. As discussed previously in the canonical case, we are thus forced to include *global* gauge transformations in  $\mathcal{G}^H$ . The space of histories  $Q^H(M)$  will now be much more complicated. However, our analysis still applies to a certain subclass of quantizations of the theory which remain in one-to-one correspondence with the IUR's of  $\pi_1(G)$  for all connected  $G$  and compact, orientable space-times  $M$ .

Those readers who are more familiar with homotopy theory may have realized that there are slicker methods for obtaining  $\pi_0(Q^H(M))$  which completely avoid the need to perform a time-slicing. (For a general discussion of these methods, along with further references, see [11].) In particular, using such techniques it can be shown that (for connected  $G$ , and  $M$  possessing a special basepoint)  $\pi_0(Q^H(M)) = H^2(\tilde{M}; \pi_1(G))$ . Here,  $\tilde{M}$  is the surface obtained by closing up the boundary circles of the compact two-dimensional space-time  $M$ , and  $H^2(\tilde{M}; \pi_1(G))$  is the second cohomology group of  $\tilde{M}$  with coefficients in  $\pi_1(G)$ . The elements of  $H^2(\tilde{M}; \pi_1(G))$  label the principal  $G$ -bundles over the closed surface  $\tilde{M}$ , and the gauge fields belonging to the path-component of  $Q^H(M)$  associated with any such element can be viewed as connections on the corresponding bundle. If  $M$  is orientable, then  $H^2(\tilde{M}; \pi_1(G)) = \pi_1(G)$  independent of the genus and the number of boundary components of  $M$ , and we recover the above results. Our main reason for performing the analysis using the more elementary time-slicing methods (and holonomies rather than gauge fields) is that we feel it makes the results more transparent by relating everything back to the cylindrical case. There is, however, something new that we can learn from the more advanced methods. More specifically, the relation  $\pi_0(Q^H(M)) = H^2(\tilde{M}; \pi_1(G))$  still holds even if  $M$  is a *nonorientable* surface (such as the projective plane, the Möbius strip, or the Klein bottle). In this case,  $H^2(\tilde{M}; \pi_1(G))$  is isomorphic to the quotient group  $\pi_1(G)/2\pi_1(G)$ , where the normal subgroup  $2\pi_1(G)$  consists of all squares of elements in  $\pi_1(G)$ . (Modding out by this normal subgroup is equivalent to adding relations to  $\pi_1(G)$  which state that every element is equal to its inverse.) The inequivalent quantizations of these theories are then labelled by the IUR's of  $\pi_1(G)/2\pi_1(G)$  independent of the details of the compact, nonorientable space-time  $M$ . This result can, in principle, also be obtained using the time-slicing methods. However, the reasoning is somewhat more subtle.

We close with two comments. First, we are currently in the process of addressing many of the questions which have already been studied in the  $\theta = 0$  case, only now for nonzero  $\theta$ . For example, can we still obtain the exact partition functions when  $\theta \neq 0$ ? Are

these theories still string theories for the appropriate gauge groups? Is there still a large- $N$  phase transition as one moves from weak to strong coupling on the space-times  $S^2$  and  $P^2$ ? The answers to these questions may provide insights into four-dimensional gauge theories in nontrivial  $\theta$ -vacua. Second, we note that our study of  $\theta$ -vacua in 2D Yang-Mills theories on an arbitrary space-time is an application of a much more general procedure for relating topology change and inequivalent quantizations which will be outlined in a forthcoming paper [12] (see also related work in [13]).

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